MUTATIONS OF FAKE WEIGHTED PROJECTIVE PLANES

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Abstract. In previous work by Coates, Galkin, and the authors, the notion of mutation between lattice polytopes was introduced. Such a mutation gives rise to a deformation between the corresponding toric varieties. In this paper we study one-step mutations that correspond to deformations between weighted projective planes, giving a complete characterisation of such mutations in terms of T-singularities. We show also that the weights involved satisfy Diophantine equations, generalising results of Hacking–Prokhorov.

1. Introduction

In [ACGK12] we described a combinatorial notion of mutation between convex lattice polytopes. In this paper we begin to explore the geometry behind this idea. Given a convex lattice polytope P containing the origin and with primitive vertices, there is a corresponding toric variety X defined by the spanning fan of P. A mutation between polytopes P and Q determines a deformation between X_P and X_Q [Ilt12]. Our main result characterises mutations between triangles; thus we characterise certain deformations, over $\mathbb{P}^1$, with fibers given by fake weighted projective planes. We recover and generalise certain results of Hacking and Prokhorov [HP10, Theorem 4.1] connecting the fake weighted projective planes with T-singularities to solutions of Markov-type equations. We prove the following:

Proposition 1.1. *Let $X = \mathbb{P}(\lambda_0, \lambda_1, \lambda_2)$ be a weighted projective plane. Up to reordering of the weights, there exists a one-step mutation to a weighted projective plane Y if and only if $\frac{1}{\lambda_0}(\lambda_1, \lambda_2)$ is a T-singularity. When this is the case, $Y = \mathbb{P}\left(\lambda_1, \lambda_2, \frac{(\lambda_1+\lambda_2)^2}{\lambda_0}\right)$. More generally, there exists a one-step mutation from the fake weighted projective plane $X/(\mathbb{Z}/n)$ to the fake weighted projective plane $Y/(\mathbb{Z}/n')$ only if $n = n'$ and $\frac{1}{\lambda_0}(\lambda_1, \lambda_2)$ is a T-singularity.*

In Proposition 3.12 we associate to a weighted projective plane X a Diophantine equation

$$mx_0x_1x_2 = k(c_0x_0^2 + c_1x_1^2 + c_2x_2^2). \tag{1.1}$$

The weights $(\lambda_0, \lambda_1, \lambda_2)$ of X correspond to a solution (a_0, a_1, a_2), where $\lambda_i = c_ia_i^2$, $i = 0, 1, 2$, and the degree of X is given by

$$(-K_X)^2 = \frac{m^2}{c_0c_1c_2k^2}.$$

One-step mutations of X correspond to transformations of the solutions to (1.1), and all such solutions can be generated from the so-called minimal weights by mutation.

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When $X = \mathbb{P}^2$, equation (1.1) becomes the celebrated Markov equation [Mar80]. Certain other special cases were studied by Rosenberger [Ros79]. These cases all have finitely many minimal weights. In §4 we give an example where the corresponding Diophantine equation has infinitely many minimal weights.

2. Mutations of Fano polytopes

Let $N \cong \mathbb{Z}^n$ be a lattice with dual $M := \mathrm{Hom}(N, \mathbb{Z})$. A lattice polytope $P \subset N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ is called *Fano* if it satisfies three conditions:

(1) P is of maximum dimension, $\dim P = \dim N$;
(2) The origin is contained in the strict interior of P, $\mathbf{0} \in \mathrm{int}(P)$;
(3) The vertices $\mathrm{vert}(P)$ of P are primitive lattice points, i.e. for any $v \in \mathrm{vert}(P)$ there are no other lattice points on the line segment $\overline{\mathbf{0}v}$ joining v and the origin.

The dual of P is defined to be the polyhedron

$$P^\vee := \{u \in M_\mathbb{Q} \mid u(v) \geqslant -1 \text{ for all } v \in P\} \subset M_\mathbb{Q}.$$

By condition (2) this is a polytope with $\mathbf{0} \in \mathrm{int}(P^\vee)$, although it need not be a lattice polytope. See [KN12] for an overview of Fano polytopes.

We briefly recall the notation of [ACGK12, §3]. Any choice of primitive vector $w \in M$ determines a lattice height function $w : N \to \mathbb{Z}$ which naturally extends to $N_\mathbb{Q} \to \mathbb{Q}$. A subset $S \subset N_\mathbb{Q}$ is said to lie at height $h \in \mathbb{Q}$ with respect to w if $w(S) := \{w(s) \mid s \in S\} = \{h\}$; we write $w(S) = h$. The set of all points of $N_\mathbb{Q}$ lying at height h with respect to a given w is an affine hyperplane $H_{w,h} := \{v \in N_\mathbb{Q} \mid w(v) = h\}$. In particular,

$$w_h(P) := \mathrm{conv}(H_{w,h} \cap P \cap N) \subset N_\mathbb{Q}$$

will denote the (possibly empty) convex hull of all lattice points in P at height h.

Define

$$h_{\min} := \min\{w(v) \mid v \in P\}, \qquad h_{\max} := \max\{w(v) \mid v \in P\}.$$

Since P is a lattice polytope, both $h_{\min}$ and $h_{\max}$ are integers. Condition (2) guarantees that $h_{\min} < 0$ and $h_{\max} > 0$.

Definition 2.1. A *factor* of P with respect to w is a lattice polytope $F \subset N_\mathbb{Q}$ satisfying:

(1) $w(F) = 0$;
(2) For every integer h, $h_{\min} \leqslant h < 0$, there exists a (possibly empty) lattice polytope $G_h \subset N_\mathbb{Q}$ at height h such that

$$H_{w,h} \cap \mathrm{vert}(P) \subseteq G_h + (-h)F \subseteq w_h(P).$$

Note that, for given polytope $P \subset N_\mathbb{Q}$ and width vector $w \in M$, a factor F need not exist. When a factor does exist we make the following construction:

Definition 2.2 ([ACGK12, Definition 5])**.** Let $P \subset N_{\mathbb{Q}}$ be a polytope with width vector $w \in M$, factor F, and polytopes $\{G_h\}$. We define the corresponding *combinatorial mutation* to be the convex lattice polytope

$$\mathrm{mut}_w(P, F; \{G_h\}) := \mathrm{conv}\left(\bigcup_{h=h_{\min}}^{-1} G_h \cup \bigcup_{h=0}^{h_{\max}} (w_h(P) + hF) \right) \subset N_{\mathbb{Q}}.$$

For brevity we will often refer to a combinatorial mutation simply as a *mutation*.

We summarise the key properties of mutations [ACGK12]:

(1) We need only consider factors F up to translation, since for any $v \in N$ such that $w(v) = 0$, we have $\mathrm{mut}_w(P, F; \{G_h\}) \cong \mathrm{mut}_w(P, v + F; \{G_h + hv\})$. In particular, choosing F to be a point leaves P unchanged.
(2) If $\{G_h\}$ and $\{G'_h\}$ are any two collections of polytopes for a factor F, then $\mathrm{mut}_w(P, F; \{G_h\}) \cong \mathrm{mut}_w(P, F; \{G'_h\})$. Thus the choice of collection $\{G_h\}$ is irrelevant and we write $\mathrm{mut}_w(P, F)$.
(3) P is a Fano polytope if and only if $\mathrm{mut}_w(P, F)$ is a Fano polytope.
(4) Let $Q := \mathrm{mut}_w(P, F)$. Then $\mathrm{mut}_{-w}(Q, F) = P$, so mutations are invertible.
(5) The toric varieties defined by the spanning fans of P and $\mathrm{mut}_w(P, F)$ have the same degree.

A mutation of $P \subset N_{\mathbb{Q}}$ induces a piecewise linear transformation φ of $M_{\mathbb{Q}}$ such that $(\varphi(P^\vee))^\vee = \mathrm{mut}_w(P, F)$, given by

$$\varphi : u \mapsto u - u_{\min} w, \qquad u \in M_{\mathbb{Q}},$$

where $u_{\min} := \min\{u(v_F) \mid v_F \in \mathrm{vert}(F)\}$. The inner normal fan of $F \subset N_{\mathbb{Q}}$ determines a chamber decomposition of $M_{\mathbb{Q}}$, and φ acts as a linear transformation on the interior of each maximal dimensional cone of this fan.

Example 2.3. Consider the triangle $P = \mathrm{conv}\{(1, -1), (-1, 2), (0, -1)\} \subset N_{\mathbb{Q}}$ corresponding to the toric variety $\mathbb{P}^2$. Let $w = (0, 1) \in M$ and set $F = \mathrm{conv}\{\mathbf{0}, (1, 0)\} \subset N_{\mathbb{Q}}$. This defines a mutation from P to the triangle $Q = \mathrm{conv}\{(1, 2), (-1, 2), (0, -1)\} \subset N_{\mathbb{Q}}$, as illustrated in Figure 1. On the dual side, this corresponds to a piecewise linear map $\varphi : u \mapsto uM_\sigma$ for $u = (\alpha, \beta) \in M_{\mathbb{Q}}$, where

$$M_\sigma = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \alpha \geqslant 0, \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

In particular, $\varphi(P^\vee) = Q^\vee$.

Mutations are particularly simple in the two-dimensional case. In this setting, a given primitive $w \in M$ defines a non-trivial mutation of $P \subset N_{\mathbb{Q}}$ if and only if $w \in \{\overline{u} \mid u \in \mathrm{vert}(P^\vee)\} \subset M$, where $\overline{u} \in M$ is the unique primitive lattice vector on the ray passing through u. Nontrivial factors $F \subset N_{\mathbb{Q}}$ are just line segments, so it suffices to restrict attention to those F which have vertex set $\{\mathbf{0}, f\}$, for some $f \in N$ with $w(f) = 0$. The inner normal fan of any factor F of P with respect to a given w is just the linear subspace of $M_{\mathbb{Q}}$ spanned by w. This divides $M_{\mathbb{Q}}$ into

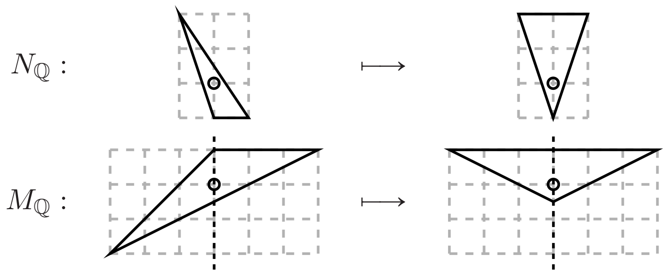


FIGURE 1. A mutation from the triangle associated with $\mathbb{P}^2$ to the triangle associated with $\mathbb{P}(1,1,4)$.

two chambers; the piecewise linear transformation φ acts trivially in one of the chambers, and as $u \mapsto u - u(f)w$ in the other.

3. One-step mutations of triangles

Set $N \cong \mathbb{Z}^2$ and let $P := \mathrm{conv}\{v_0, v_1, v_2\} \subset N_{\mathbb{Q}}$ be a Fano triangle. Since $\mathbf{0} \in \mathrm{int}(P)$ there exists a (unique) choice of coprime positive integers $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}_{>0}$ with $\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 = \mathbf{0}$. The projective toric surface X given by the spanning fan of P has Picard rank 1, and is called a *fake weighted projective plane* with weights $(\lambda_0, \lambda_1, \lambda_2)$; X is the quotient of $\mathbb{P}(\lambda_0, \lambda_1, \lambda_2)$ by the action of a finite group of order $\mathrm{mult}(X)$ acting free in codimension one [Buc08, Kas09].

Remark 3.1. Since the vertices of P are primitive, the weights $(\lambda_0, \lambda_1, \lambda_2)$ are *well-formed*: that is, $\gcd\{\lambda_i, \lambda_j\} = 1$, $i \neq j$. In this paper we will always require that weights are well-formed.

Definition 3.2. We say that a fake weighted projective plane Y with defining Fano triangle $Q \subset N_{\mathbb{Q}}$ is obtained from X by a *one-step mutation* if $Q \cong \mathrm{mut}_w(P, F)$ for some choice of w and factor F.

3.1. One-step mutations in $M_{\mathbb{Q}}$ and weights.

Proposition 3.3. *Let X be a fake weighted projective plane with weights $(\lambda_0, \lambda_1, \lambda_2)$. Suppose there exists a one-step mutation to a fake weighted projective plane Y. Then, up to relabelling, $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$ and Y has weights*

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right).$$

Proof. Consider a lattice triangle $T_1 \subset N_{\mathbb{Q}}$, $\mathbf{0} \in \mathrm{int}(T_1)$, and suppose that there exists width vector $w \in M$ and factor $F \subset N_{\mathbb{Q}}$, $w(F) = 0$, such that the mutation $T_2 = \mathrm{mut}_w(T_1, F)$ is also a triangle. Without loss of generality we can assume that $w = (0,1) \in M$ and $F = \mathrm{conv}\{\mathbf{0}, (a,0)\}$ for some $a \in \mathbb{Z}_{>0}$. The mutation corresponds to a piecewise linear action on $M_{\mathbb{Q}}$ via $u \mapsto uM_\sigma$

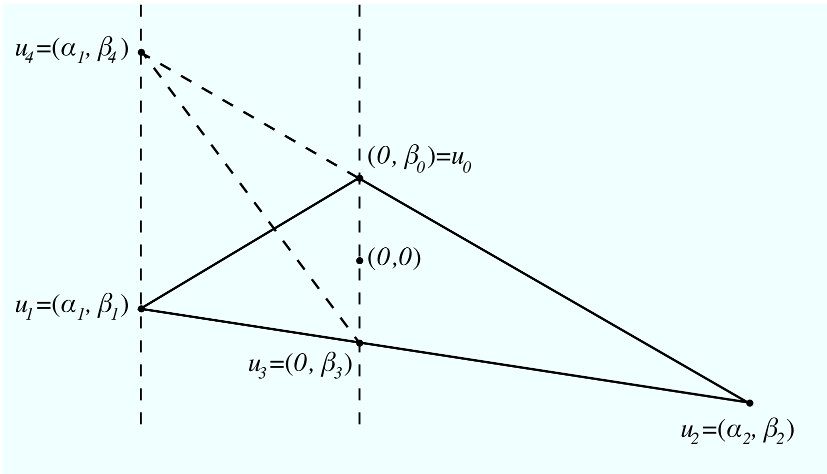


FIGURE 2. A one-step mutation, depicted in $M_{\mathbb{Q}}$, of the triangle $\mathrm{conv}\{u_0, u_1, u_2\}$ to the triangle $\mathrm{conv}\{u_2, u_3, u_4\}$.

given by

$$M_\sigma = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } u \in M^+, \\ \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

where M^+ is the half-space $\{(\alpha, \beta) \in M_{\mathbb{Q}} \mid \alpha > 0\}$. Let $T_1{}^\vee = \mathrm{conv}\{u_0, u_1, u_2\} \subset M_{\mathbb{Q}}$ be the (possibly rational) triangle dual to T_1, where $u_2 \in M^+$ and so is fixed under the action of the mutation, and $u_1 \in M^- := \{(\alpha, \beta) \in M_{\mathbb{Q}} \mid \alpha < 0\}$. Since $T_2{}^\vee \subset M_{\mathbb{Q}}$ is also a triangle, the only possibility is that u_0 lies on the line $\langle w \rangle := \{\gamma w \in M_{\mathbb{Q}} \mid \gamma \in \mathbb{Q}\}$, $T_2{}^\vee = \mathrm{conv}\{u_2, u_3, u_4\}$ where u_0 is contained in the line segment $\overline{u_2 u_4}$ joining u_2 and u_4, and u_3 is contained in the line segment $\overline{u_1 u_2}$. This situation is illustrated in Figure 2.

Since $\mathbf{0} \in T_1{}^\vee$ there exist unique weights $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$, $\gcd\{\lambda_0, \lambda_1, \lambda_2\} = 1$, such that

$$\lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2 = \mathbf{0}. \tag{3.1}$$

Since $u_3 = (0, \beta_3) \in \overline{u_1 u_2}$ there exists some $0 < \mu < 1$ such that $\mu\alpha_1 + (1 - \mu)\alpha_2 = 0$. But $\lambda_1\alpha_1 + \lambda_2\alpha_2 = 0$, hence

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}\alpha_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}\alpha_2 = 0.$$

By uniqueness of μ,

$$u_3 = \frac{\lambda_1}{\lambda_1 + \lambda_2}u_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}u_2. \tag{3.2}$$

Similarly, since $u_0 = (0, \beta_0) \in \overline{u_2 u_4}$ there exists some $0 < \nu < 1$ such that $u_0 = \nu u_2 + (1-\nu)u_4$, giving

$$u_4 = \frac{1}{1 - \nu}u_0 - \frac{\nu}{1 - \nu}u_2.$$

Comparing coefficients we see that

$$\alpha_1 = -\frac{\nu}{1-\nu}\alpha_2. \tag{3.3}$$

But $u_4 = u_1 + \kappa u_0$ for some $\kappa > 0$. Combining this with equation (3.1) we see that

$$u_4 = \frac{\lambda_1\kappa - \lambda_0}{\lambda_1}u_0 - \frac{\lambda_2}{\lambda_1}u_2.$$

Comparing coefficients, we obtain

$$\alpha_1 = -\frac{\lambda_2}{\lambda_1}\alpha_2. \tag{3.4}$$

Equating equations (3.3) and (3.4) gives

$$u_4 = \frac{\lambda_1 + \lambda_2}{\lambda_1}u_0 - \frac{\lambda_2}{\lambda_1}u_2. \tag{3.5}$$

Notice that, since both u_0 and u_3 are contained in $\langle w\rangle$, there exists some $\gamma > 0$ such that $-\gamma u_3 = u_0$. Substituting into equation (3.5) we have

$$\frac{\lambda_2}{\lambda_1}u_2 + u_4 + \gamma' u_3 = \mathbf{0} \tag{3.6}$$

where $\gamma' = \gamma(\lambda_1 + \lambda_2)/\lambda_1 > 0$. Substituting in equation (3.2) we obtain

$$\frac{\lambda_2}{\lambda_1}u_2 + u_4 + \frac{\gamma'\lambda_1}{\lambda_1+\lambda_2}u_1 + \frac{\gamma'\lambda_2}{\lambda_1+\lambda_2}u_2 = \mathbf{0}.$$

Using equation (3.5) to rewrite the first two terms and clearing denominators gives:

$$(\lambda_1+\lambda_2)^2 u_0 + \gamma'\lambda_1^2 u_1 + \gamma'\lambda_1\lambda_2 u_2 = \mathbf{0}. \tag{3.7}$$

Set $h := \lambda_0 + \lambda_1 + \lambda_2$ and $\Gamma := (\lambda_1+\lambda_2)^2 + \gamma'\lambda_1^2 + \gamma'\lambda_1\lambda_2$. By comparing equations (3.1) and (3.7), uniqueness of barycentric coordinates gives:

$$\begin{aligned} h(\lambda_1+\lambda_2)^2 &= \Gamma\lambda_0,\\ h\gamma'\lambda_1^2 &= \Gamma\lambda_1,\\ h\gamma'\lambda_1\lambda_2 &= \Gamma\lambda_2. \end{aligned}$$

In particular,

$$\gamma' = \frac{(\lambda_1+\lambda_2)^2}{\lambda_0\lambda_1}.$$

Substituting this expression for γ' back into equation (3.6) gives

$$\lambda_0\lambda_2 u_2 + (\lambda_1+\lambda_2)^2 u_3 + \lambda_0\lambda_1 u_4 = \mathbf{0}. \tag{3.8}$$

Finally, we consider the situation where $T_1 \subset N_{\mathbb{Q}}$ is the triangle associated with a fake weighted projective plane with weights $(\lambda_0, \lambda_1, \lambda_2)$, and assume that there exists a one-step mutation to some triangle $T_2 \subset N_{\mathbb{Q}}$. If λ_0 does not divide $(\lambda_1+\lambda_2)^2$, then by equation (3.8) the associated weights are

$$\left(\lambda_0\lambda_1, \lambda_0\lambda_2, (\lambda_1+\lambda_2)^2\right),$$

and these fail to be well-formed when $\lambda_0 > 1$. Therefore, we must have $\lambda_0 \mid (\lambda_1+\lambda_2)^2$, giving weights

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1+\lambda_2)^2}{\lambda_0}\right).$$

□

Remark 3.4. Let $(\lambda_0, \lambda_1, \lambda_2)$ be well-formed weights such that $\lambda_0 \mid (\lambda_1+\lambda_2)^2$. Then

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1+\lambda_2)^2}{\lambda_0}\right)$$

are also well-formed. For suppose there exists some prime p such that

$$p \mid \lambda_1 \quad \text{and} \quad p \mid \frac{(\lambda_1+\lambda_2)^2}{\lambda_0}.$$

Then $p \mid \lambda_2^2$ and so $p \mid \lambda_2$. But this contradicts $(\lambda_0, \lambda_1, \lambda_2)$ being well-formed.

Example 3.5. There exists no one-step mutation from $\mathbb{P}(3,5,11)$ to any other weighted projective space, since $3 \nmid (5+11)^2$, $5 \nmid (3+11)^2$, and $11 \nmid (3+5)^2$.

Example 3.6. The requirement that $\lambda_0 \mid (\lambda_1+\lambda_2)^2$ in Proposition 3.3 is necessary but not sufficient. For example, consider the triangle $T = \mathrm{conv}\{(10,-7),(-5,2),(0,1)\} \subset N_{\mathbb{Q}}$. This has weights $(1,2,3)$, however there exist no one-step mutations from T.

3.2. One-step mutations in $N_{\mathbb{Q}}$ and T-singularities.

Definition 3.7 ([KSB88, Definition 3.7])**.** A quotient surface singularity is called a *T-singularity* if it admits a $\mathbb{Q}$-Gorenstein one-parameter smoothing.

T-singularities include the du Val singularities $\frac{1}{r}(1, r-1)$, and are cyclic quotient singularities of the form $\frac{1}{nd^2}(1, dna-1)$, where $\gcd\{d,a\} = 1$ [KSB88, Proposition 3.10].

Lemma 3.8. *An isolated quotient singularity $\frac{1}{r}(a,b)$ is a T-singularity if and only if $r \mid (a+b)^2$.*

Proof. We begin by noting that the condition that $r \mid (a+b)^2$ is independent of the choice of representation of $\frac{1}{r}(a,b)$. For let c be any integer coprime to r. Then $r \mid (a+b)^2$ if and only if $r \mid c^2(a+b)^2 = (ca+cb)^2$.

Suppose we are given a T-singularity. Writing the singularity in the form $\frac{1}{nd^2}(1, dna-1)$ where $\gcd\{d,a\} = 1$, we see that $nd^2 \mid d^2n^2a^2$. Conversely consider the isolated quotient singularity $\frac{1}{r}(a,b)$. Since a is invertible $\bmod\ r$, we can write this as $\frac{1}{r}(1, b'-1)$, where $b' \equiv ba^{-1}+1 \pmod{r}$. Write $r = nd^2$ where n is square-free. Since $nd^2 \mid b'^2$ by assumption, we see that $nd \mid b'$. In particular, we can express our singularity in the form $\frac{1}{nd^2}(1, dn\alpha-1)$ for some $\alpha \in \mathbb{Z}_{>0}$. Finally, we note that this really is a T-singularity: if $\gcd\{d,\alpha\} = c$ then we can absorb this factor into $n' = nc^2$ whilst rescaling $d' = d/c$ and $\alpha' = \alpha/c$. □

Proposition 3.9. *Let X be a fake weighted projective plane corresponding to a triangle $T \subset N_{\mathbb{Q}}$, and suppose that the cone C spanned by an edge E of T corresponds to a $\frac{1}{r}(a,b)$ singularity. There exists a one-step mutation to a fake weighted projective plane Y given by* $\mathrm{mut}_w(T,F)$ *with $w(E) = h_{\min}$ if and only if $\frac{1}{r}(a,b)$ is a T-singularity.*

Proof. Let X correspond to the lattice triangle $T = \mathrm{conv}\{v_1, v_2, v_3\} \subset N_{\mathbb{Q}}$, where $\mathbf{0} \in \mathrm{int}(T)$ and the vertices $\mathrm{vert}(T) \subset N$ are all primitive. Consider the cone $C = \mathrm{cone}\{v_1, v_2\}$ spanned by the edge $E = \overline{v_1 v_2}$; this is an isolated quotient singularity (possibly smooth), so is of the form $\frac{1}{r}(a,b)$ for some $r, a, b \in \mathbb{Z}_{>0}$, $\gcd\{r,a\} = \gcd\{r,b\} = 1$.

Let $w \in M$ be a primitive lattice point such that $w(v_1) = w(v_2) = h$ for some $h < 0$. Then, up to translation, there exists a factor $F \subset N_{\mathbb{Q}}$, $w(F) = 0$, such that $T' := \mathrm{mut}_w(T,F)$ is a triangle if and only if $v_1 + (-h)F = E$. Equivalently, if and only if $h \mid |E \cap N| - 1$.

Finally, we express the values of h and $|E \cap N| - 1$ in terms of the singularity $\frac{1}{r}(a,b)$. Set $k := \gcd\{r, a+b\}$. Then the height $h = -r/k$, and the number of points on the edge E is given by

$$|\{m \mid m \in \{0, \ldots, r\} \text{ and } (a+b)m \equiv 0 \pmod r\}| = 1 + \frac{r}{h} = 1 + k.$$

Hence $h \mid |E \cap N| - 1$ if and only if $r/k \mid k$. But $r/k \mid k$ if and only if $r \mid \gcd\{r, a+b\}^2 = \gcd\{r^2, (a+b)^2\}$, and $r \mid \gcd\{r^2, (a+b)^2\}$ if and only if $r \mid (a+b)^2$. The result follows by Lemma 3.8. □

Example 3.10. Returning to Example 3.6, we see that the corresponding fake weighted projective space X is a quotient of $\mathbb{P}(1,2,3)$ with $\mathrm{mult}(X) = 5$. The three singularities are $\frac{1}{5}(1,3)$, $\frac{1}{10}(1,3)$, and $\frac{1}{15}(1,11)$, none of which is a T-singularity.

When X is a weighted projective plane, Proposition 3.9 tells us that the condition that $\lambda_0 \mid (\lambda_1 + \lambda_2)^2$ in Proposition 3.3 is both necessary and sufficient.

3.3. One-step mutations and Diophantine equations.

Lemma 3.11. *Let $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}^3_{>0}$ with $d = \gcd\{\lambda_0, \lambda_1, \lambda_2\}$. Write:*

1. *$\lambda_i = d c_i a_i^2$, where $a_i, c_i \in \mathbb{Z}_{>0}$ and c_i is square-free;*
2. *$(\lambda_0 + \lambda_1 + \lambda_2)^2/(\lambda_0\lambda_1\lambda_2) = m^2/(rk^2)$, where $m, k, r \in \mathbb{Z}_{>0}$ and r is square-free;*
3. *$c_0 c_1 c_2 = gS^2$ and $dr = hT^2$, where $g, h, S, T \in \mathbb{Z}_{>0}$ and both g and h are square-free.*

Then (da_0, da_1, da_2) is a solution to the Diophantine equation

$$Smx_0x_1x_2 = Tk(c_0x_0^2 + c_1x_1^2 + c_2x_2^2). \tag{3.9}$$

Proof. By substituting expressions (1) and (3) into (2) we obtain

$$gS^2m^2(da_0)^2(da_1)^2(da_2)^2 = hT^2k^2\left(c_0(da_0)^2 + c_1(da_1)^2 + c_2(da_2)^2\right)^2.$$

Comparing square-free parts, we conclude that $g = h$. Cancelling and taking square-roots on both sides establishes the result. □

Since the weights are assumed to be well-formed, $d = S = T = 1$ and equation (3.9) becomes

$$mx_0x_1x_2 = k(c_0x_0^2 + c_1x_1^2 + c_2x_2^2). \tag{3.10}$$

Suppose that (a_0, a_1, a_2) is a positive integral solution to equation (3.10), so that $\lambda_i = c_i a_i^2$. The expression

$$\frac{(\lambda_0 + \lambda_1 + \lambda_2)^2}{\lambda_0\lambda_1\lambda_2} \tag{3.11}$$

occurring in Lemma 3.11 is equal to the degree of $\mathbb{P}(\lambda_0, \lambda_1, \lambda_2)$. More generally if X is a fake weighted projective plane with weights $(\lambda_0, \lambda_1, \lambda_2)$ then (3.11) is equal to $\mathrm{mult}(X)(-K_X)^2$.

Proposition 3.12. *Let X be a fake weighted projective plane and suppose that there exists a one-step mutation to a fake weighted projective plane Y. Then the weights of X and Y give solutions to the same Diophantine equation* (3.10)*. In particular,* $\mathrm{mult}(X) = \mathrm{mult}(Y)$.

Proof. With notation as in Lemma 3.11, we can write the weights $(\lambda_0, \lambda_1, \lambda_2)$ of X in the form $\lambda_i = c_i a_i^2$, where the c_i are square-free positive integers. From Proposition 3.3 we know that Y has weights

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right) = \left(c_1 a_1^2, c_2 a_2^2, \frac{(c_1 a_1^2 + c_2 a_2^2)^2}{c_0 a_0^2}\right).$$

The final weight is an integer; in particular, it has square-free part c_0. Thus the c_i are invariant under mutation. Furthermore,

$$\begin{aligned}\frac{\left(\lambda_1 + \lambda_2 + \frac{(\lambda_1+\lambda_2)^2}{\lambda_0}\right)^2}{\lambda_1 \cdot \lambda_2 \cdot \frac{(\lambda_1+\lambda_2)^2}{\lambda_0}} &= \frac{\left(\lambda_0\lambda_1 + \lambda_0\lambda_2 + (\lambda_1 + \lambda_2)^2\right)^2}{\lambda_0\lambda_1\lambda_2(\lambda_1 + \lambda_2)^2} \\ &= \frac{(\lambda_0 + \lambda_1 + \lambda_2)^2}{\lambda_0\lambda_1\lambda_2} \\ &= \frac{m^2}{rk^2}\end{aligned}$$

and so the ratio m/k is also preserved by mutation. Hence the weights of X and of Y both generate solutions to the same Diophantine equation (3.10).

Finally we recall that degree is fixed under mutation, hence $(-K_X)^2 = (-K_Y)^2$. But

$$\frac{m^2}{rk^2} = \mathrm{mult}(X)(-K_X)^2 = \mathrm{mult}(Y)(-K_Y)^2$$

and so $\mathrm{mult}(X) = \mathrm{mult}(Y)$. □

By combining Propositions 3.3, 3.9, and 3.12 we obtain Proposition 1.1.

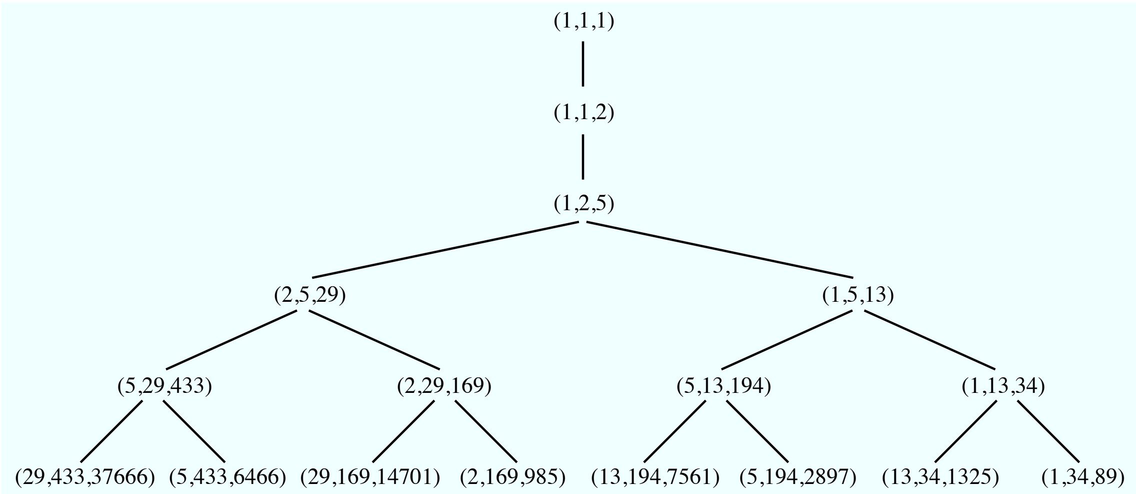
Remark 3.13. The weights of a fake weighted projective plane correspond to a solution (a_0, a_1, a_2) of equation (3.10). A one-step mutation gives a second solution via the transformation:

$$(a_0, a_1, a_2) \mapsto \left(\frac{m}{k}\frac{a_1 a_2}{c_0} - a_0, a_1, a_2\right).$$

Example 3.14. Consider $\mathbb{P}^2$. In this case $m/k = 3$, $c_0 = c_1 = c_2 = 1$, and $(1,1,1) \in \mathbb{Z}^3_{>0}$ is a solution of

$$3x_0x_1x_2 = x_0^2 + x_1^2 + x_2^2. \tag{3.12}$$

Up to isomorphism, there is a single one-step mutation to $\mathbb{P}(1,1,4)$, giving a solution $(1,1,2) \in \mathbb{Z}^3_{>0}$ of equation (3.12). Proceeding in this fashion we obtain a graph of one-step mutations corresponding to solutions of (3.12), which we illustrate to a depth of five mutations:



Definition 3.15. The *height* of the weights $(\lambda_0, \lambda_1, \lambda_2)$ is given by the sum $h := \lambda_0 + \lambda_1 + \lambda_2 \in \mathbb{Z}_{>0}$. We call the weights *minimal* if for any sequence of one-step mutations $(\lambda_0, \lambda_1, \lambda_2) \mapsto \ldots \mapsto (\lambda_0', \lambda_1', \lambda_2')$ we have that $h \leqslant h'$.

Lemma 3.16. *Given weights $(\lambda_0, \lambda_1, \lambda_2)$ at height h there exists at most one one-step mutation such that $h' \leqslant h$. Moreover, if $h' = h$ then the weights are the same.*

Proof. Without loss of generality suppose we have two one-step mutations

$$\left(\lambda_1, \lambda_2, \frac{(\lambda_1 + \lambda_2)^2}{\lambda_0}\right) \qquad \text{and} \qquad \left(\lambda_0, \frac{(\lambda_0 + \lambda_2)^2}{\lambda_1}, \lambda_2\right)$$

with respective heights h' and h'' such that $h' \leqslant h$ and $h'' \leqslant h$. Since $h' \leqslant h$ we obtain $(\lambda_1 + \lambda_2)^2 \leqslant \lambda_0^2$, and so

$$\lambda_1^2 + \lambda_2^2 < \lambda_0^2. \tag{3.13}$$

From $h'' \leqslant h$ we obtain

$$\lambda_0^2 + \lambda_2^2 < \lambda_1^2. \tag{3.14}$$

Combining equations (3.13) and (3.14) gives a contradiction, hence there exists at most one one-step mutation such that $h' \leqslant h$. If we suppose that $h' = h$ then

$$\frac{(\lambda_1 + \lambda_2)^2}{\lambda_0} = \lambda_0$$

and equality of the weights is immediate. $\square$

The height imposes a natural direction on the graph of all one-step mutations generated by the weight $(\lambda_0, \lambda_1, \lambda_2)$. Lemma 3.16 tells us that this directed graph is a tree, with a uniquely defined minimal weight.

4. Example: An infinite number of minimal weights

In this section we shall focus on the Diophantine equation

$$12x_0x_1x_2 = 3x_0^2 + 5x_1^2 + 7x_2^2. \tag{4.1}$$

Any solution (a_0, a_1, a_2) such that $(3a_0^2, 5a_1^2, 7a_2^2)$ is well-formed corresponds to weighted projective space $\mathbb{P}(3a_0^2, 5a_1^2, 7a_2^2)$ of degree $144/105$. One possible such solution is $(2, 1, 1)$ giving $\mathbb{P}(12, 5, 7)$. Consider the graph $\mathcal{G}$ of all such solutions. Two solutions lie in the same component if and only if there exists a sequence of one-step mutations between the corresponding weighted projective planes. Furthermore, each component is a tree with unique minimal weight. We shall show that there exists an infinite number of components, and that every component contains at most two solutions; in fact the only component with a single solution is $(2, 1, 1)$.

4.1. **Coprime solutions give well-formed weights.** Let (a_0, a_1, a_2) be a solution of equation (4.1) such that $\gcd\{a_0, a_1, a_2\} = 1$. Clearly this is a necessary condition for the corresponding weights $(3a_0^2, 5a_1^2, 7a_2^2)$ to be well-formed. We shall show that it is sufficient. For suppose that there exists some prime p such that $p \mid c_ia_i^2$ and $p \mid c_ja_j^2$, $i \neq j$. Since p cannot simultaneously divide both c_i and c_j, we have that p must divide either a_i or a_j. In particular, $p \mid 12a_0a_1a_2$ and so, by equation (4.1), p divides the remaining weight $c_ka_k^2$. Similarly, since p can divide at most one of 3, 5, and 7 we see that $p^2 \mid 12a_0a_1a_2$ and so p^2 divides each of the three weights. We conclude that $p \mid \gcd\{a_0, a_1, a_2\}$, contradicting coprimality.

4.2. **A necessary and sufficient condition for rational solutions when a_1 and a_2 are fixed.** Fix $a_1, a_2 \in \mathbb{Z}_{>0}$ and consider the quadratic

$$12xa_1a_2 = 3x^2 + 5a_1^2 + 7a_2^2. \tag{4.2}$$

The discriminant is given by

$$12^2a_1^2a_2^2 - 12(5a_1^2 + 7a_2^2) = 12\left(5a_1^2(a_2^2 - 1) + 7a_2^2(a_1^2 - 1)\right),$$

which is always non-negative. The discriminant is zero only in the case $a_1 = a_2 = 1$, corresponding to the solution $(2, 1, 1)$ of equation (4.1). Furthermore, we see that a rational solution to equation (4.2) exists if and only if

$$5a_1^2(a_2^2 - 1) + 7a_2^2(a_1^2 - 1) = 3N^2, \qquad \text{for some } N \in \mathbb{Z}_{>0}. \tag{4.3}$$

4.3. **Any rational solution is an integral solution.** Suppose that $\alpha, \beta \in \mathbb{R}$ are the two solutions of equation (4.2). We obtain:

$$\alpha + \beta = 4a_1a_2, \tag{4.4}$$

$$3\alpha\beta = 5a_1^2 + 7a_2^2. \tag{4.5}$$

In particular, since the right-hand side in each case is a strictly positive integer, we see that $\alpha, \beta > 0$. Furthermore, α is rational if and only if β is rational. Since we are only interested in rational solutions, we can assume that both α and β are rational. Let us write

$$\alpha = \frac{n_1}{m_1} \qquad \text{and} \qquad \beta = \frac{n_2}{m_2},$$

where the fractions are expressed in their reduced form, i.e. $\gcd\{n_i, m_i\} = 1$. Then

$$m_1m_2 \mid 3n_1n_2, \tag{4.6}$$

$$m_1m_2 \mid n_1m_2 + n_2m_1. \tag{4.7}$$

By (4.7), $m_2 \mid m_1$ and $m_1 \mid m_2$, forcing $m_1 = m_2$. Without loss of generality, from (4.6) we may assume that $m_1 \mid 3n_2$ and $m_2 \mid n_1$. But then $m_1 \mid n_1$, forcing $m_1 = m_2 = 1$. Hence $\alpha, \beta \in \mathbb{Z}_{>0}$.

4.4. **The values a_1 and a_2 are fixed under one-step mutations.** We now show that, given a solution (a_0, a_1, a_2) such that $\gcd\{a_0, a_1, a_2\} = 1$, the values of a_1 and a_2 are fixed under one-step mutation. For suppose that

$$\frac{(3a_0^2 + 7a_2^2)^2}{5a_1^2} \in \mathbb{Z}. \tag{4.8}$$

Without loss of generality we may take $\alpha = a_0$. We see that $5 \mid 3a_0^2 + 7a_2^2 = 3\alpha^2 + 3\alpha\beta - 5a_1^2$ by (4.5), hence $5 \mid 3\alpha(\alpha + \beta) = 12a_0a_1a_2$ by (4.4). Since the weights are pairwise coprime, the only possibility is that $5 \mid a_1$. Returning to equation (4.8) we see that $5^2 \mid 3a_0^2 + 7a_2^2$, and proceeding as before we find that $5^2 \mid a_1$. Clearly we can repeat this process an arbitrary number of times, increasing the power of 5 at each step. This is a contradiction. The case when

$$\frac{(3a_0^2 + 5a_1^2)^2}{7a_2^2} \in \mathbb{Z}$$

is dealt with similarly.

4.5. **An infinite number of components.** Set $a_1 = 1$ in condition (4.3). The condition becomes $a_2^2 - 1 = 15M^2$, where $5M = N$. This is a Pell equation, and Emerson [Eme69] has shown that there exists an infinite number of integer solutions given by a recurrence relation. In this case we see that $a_2^{(n)}$ and $M^{(n)}$ are generated by:

$$\begin{aligned} a_2^{(0)} &= 1, & M^{(0)} &= 0, \\ a_2^{(1)} &= 4, & M^{(1)} &= 1, \\ a_2^{(n+1)} &= 8a_2^{(n)} - a_2^{(n-1)}, & M^{(n+1)} &= 8M^{(n)} - M^{(n-1)}. \end{aligned}$$

Substituting these expressions back into the original quadratic (4.2) gives:

$$a_0^{(n+1)} = 2a_2^{(n)} \pm 5M^{(n)}.$$

These solutions are coprime (since $a_1 = 1$) and so correspond to well-formed weights. We will focus on the smaller of the two solutions, corresponding to the minimum of the two weights. Substituting the expressions for $a_2^{(n)}$ and $M^{(n)}$ gives:

$$\begin{aligned} a_0^{(n+1)} &= 2a_2^{(n+1)} - 5M^{(n+1)} \\ &= 8\left(2a_2^{(n)} - 5M^{(n)}\right) - \left(2a_2^{(n-1)} - 5M^{(n-1)}\right) \\ &= 8a_0^{(n)} - a_0^{(n-1)}. \end{aligned}$$

Hence we obtain the recurrence relation:

$$a_0^{(0)} = 2,$$
$$a_0^{(1)} = 3,$$
$$a_0^{(n+1)} = 8a_0^{(n)} - a_0^{(n-1)}.$$

Remark 4.1. If instead we insist that $a_2 = 1$, we obtain the Pell equation $a_1^2 - 1 = 21M^2$, where $7M = N$. In this case the recurrence relation is given by:

$$a_1^{(0)} = 1, \qquad M^{(0)} = 0,$$
$$a_1^{(1)} = 55, \qquad M^{(1)} = 12,$$
$$a_1^{(n+1)} = 110a_1^{(n)} - a_1^{(n-1)}, \qquad M^{(n+1)} = 110M^{(n)} - M^{(n-1)}.$$

Proceeding as above we find that

$$a_0^{(0)} = 2,$$
$$a_0^{(1)} = 26,$$
$$a_0^{(n+1)} = 110a_0^{(n)} - a_0^{(n-1)}.$$

Hence we have a second infinite family of components of $\mathcal{G}$. Notice that these two families do not exhaust all the possibilities: for example, $a_1 = 5$, $a_2 = 4$ satisfies condition (4.3), giving the two solutions $(1, 5, 4)$ and $(79, 5, 4)$.

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